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The polarizability of chains of touching cylinder pairs

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Abstract. By solving the Laplace equation in an appropriate coordinate frame, simple expressions have been derived for the polarizability per unit area of infinitely long chains of touching pairs of cylinders. The method used presupposes no restrictions on the value of the dielectric constant of the cylindrical particles and yields an exact analytic solution incorporating all multipoles. The results offer an insight into the effect of long range interactions between particle pairs. Moreover the technique can be extended to the study of arrays generally. Comparisons are made between the induced dipole moment on pairs within a chain and the moment on an isolated touching pair in the case of aluminium columns in air. A quasistatic assumption is used at optical frequencies. It is found that while the general behaviour of the wavelength-dependent polarizability is the same in both models, there are significant differences in the average degree of absorption and polarization.

1. Introduction

The study of the electromagnetic response to an applied electric field of arrays of small particles embedded in another medium has a long history. It remains of interest for both fundamental and technical reasons. The widely used dipole theories referred to as Clausius-Mosotti, Lorenz-Lorentz or Maxwell-Garnett (depending on the frequency domain) and related effective medium models such as the Bruggeman model are not able to describe the impact of close approach between particles. However, close approach commonly occurs and can lead to substantial changes in response. While there has been some useful progress in this area based on the techniques proposed originally by Lord Rayleigh [1] which require structure factor sums and very large matrices, few useful simple expressions have been found. These are needed for better understanding of real systems and for routine analysis of experimental data. This report is part of a programme to provide useful analytic expressions for such systems.

In an earlier work based on a conformal mapping technique [2] study was made of the degree of optical absorption and reflection of a thin film with fine columnar metal inclusions. The sizes of and spacing between the metal columns is small enough for a static approximation to apply. The dependence of absorptance and reflectance on the wavelength of the applied field was calculated successively for a film with separate, touching and intersecting cylinder pairs as the basic metal inclusion. The model for polarizability considered each of these pairs in isolation. The long-range effects between pairs, being weak, could be assumed treatable within a Clausius-Mosotti type of approximation. Of the three cases considered it was found that maximum absorption occurs for a *touching* pair of cylinders.

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Recent work on the transport properties of composite media [3–9] has provided asymptotic expansions and perturbation formulae for various chains and arrays of cylindrical inclusions. In the present study a generalization of the case of touching cylinder pairs will be developed. By working in a new coordinate frame we shall find *exact analytic expressions* for the polarizability of infinite chains of touching cylinder pairs. The cylinder pairs may be aligned along (chain 1) or perpendicular to (chain 2) the chain axis (see figures 1(a) and 1(c) respectively). The expressions for the polarizabilities of the two chains, as well as their derivations, are analogous. Hence the derivations of the final expressions for chain 1 will be presented in detail whilst those for chain 2 will merely be stated.

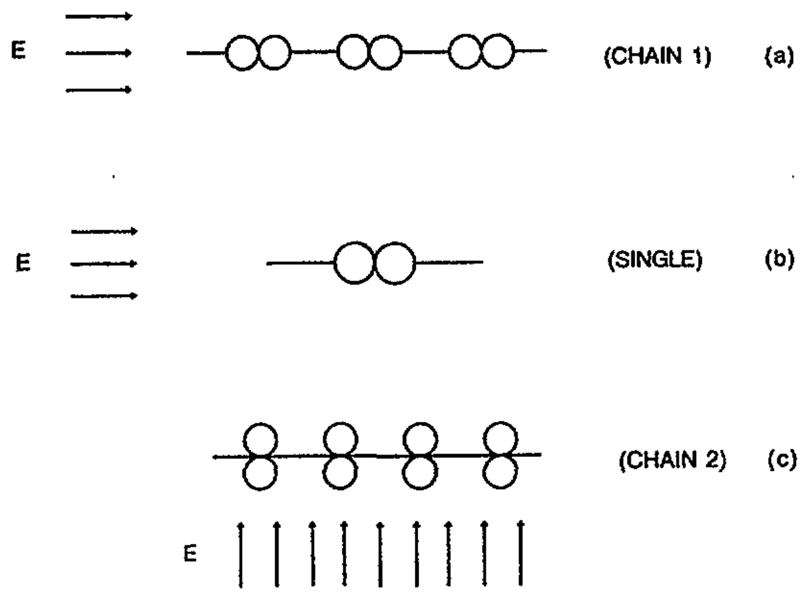

Figure 1. (a) Chain 1 with an applied x-field, (b) a single isolated pair of touching unit circles with an applied x-field and (c) chain 2 with an applied y-field.

Using a conformal mapping method we shall create a coordinate frame in which the particle boundary is specified by a constant value of one of the variables. This makes it possible to separate variables in the boundary conditions. In this transformed frame we then find the solution of the electrostatic problem by solving the Laplace equation for static fields applied longitudinally and transversally. The series expansion of the potential external to the particles along the Cartesian axes is determined. The $1/r$ coefficient in these expansions is found using residue theory. The resulting expressions, which represent the polarizabilities as series, are analytically continued and as a result are re-expressed in integral forms valid for all values of the complex dielectric contrast lying in the cut plane.

2. The longitudinal and transverse polarizabilities for chain 1

We shall adopt the following conventions throughout. External to the cylinders (region 1), that is, in the film matrix, the dielectric constant is 1, whilst within the cylinders (region 2) it is ε. For simplicity the strength of the applied field will be taken to be unity. In this case the polarizability (α) and the dipole moment (p) for each particle will be numerically equal since $p = \alpha E$.

2.1. The transformation

We now find a conformal mapping which takes us from the (x, y) frame to the (u, v) frame. We begin by letting

$$z = x + \mathrm{i}y \qquad w = u + \mathrm{i}v$$

and seek our transformation in the form

$$w = \mathscr{F}(\bar{z})$$

where the particle boundary is to be given by a constant value of u. Now, the transformation in which constant $|u|$ represents a single pair of horizontally aligned touching circles tangent at the origin of the z plane is [10]

$$w_0 = \frac{1}{\bar{z}}. \tag{1}$$

The closed curves given by constant u and constant v from (1) are in fact precisely the equipotentials and force lines respectively resulting from a dipole at the origin of the z plane. Hence if we place dipoles w_n at the points $z = na$, $n \in \boldsymbol{Z}$, we find that the required transformation will be given by

$$w = \sum_n w_n$$

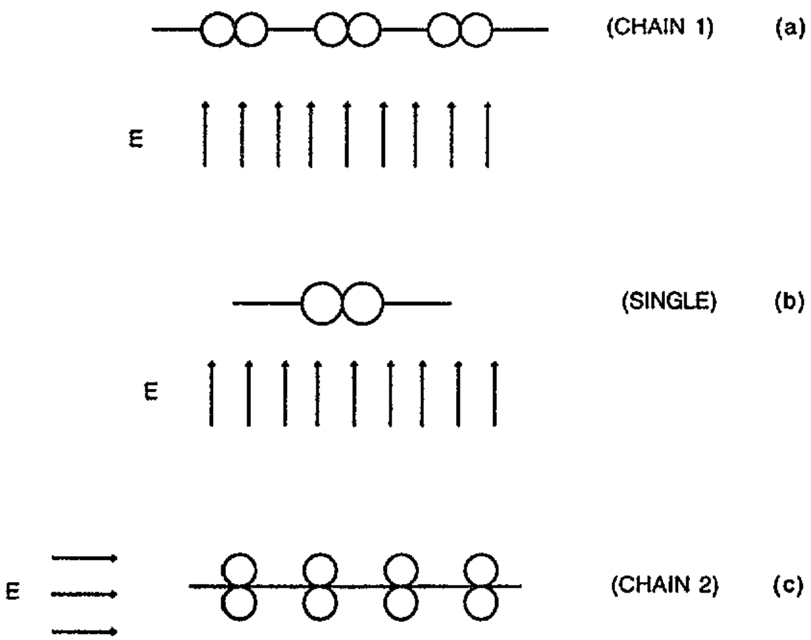

Figure 2. (a) Chain 1 with an applied y-field, (b) a single isolated pair of touching unit circles with an applied y-field and (c) chain 2 with an applied x-field.

that is

$$w=\sum_{n=-\infty}^{\infty}\frac{1}{\bar z-na}=\frac{1}{\bar z}+\sum_{n=1}^{\infty}\frac{1}{\bar z-na}+\sum_{n=1}^{\infty}\frac{1}{\bar z+na}=\frac{1}{\bar z}+2\bar z\sum_{n=1}^{\infty}\frac{1}{\bar z^2-n^2a^2}.$$

This last expression is precisely the partial fraction expansion of $(\pi/a)\cot(\pi\bar z/a)$ [11], and so the transformation we seek is given by

$$w=\sigma\cot\sigma\bar z \qquad \sigma=\frac{\pi}{a}. \tag{2}$$

The distance between the pairs' points of tangency is a, and as $a\to\infty(\sigma\to 0)$ we recover the transformation (1). When split into its real and imaginary parts (2) yields

$$u=\frac{\sigma\sin 2\sigma x}{2(\sin^2\sigma x+\sinh^2\sigma y)} \qquad v=\frac{\sigma\sinh 2\sigma y}{2(\sin^2\sigma x+\sinh^2\sigma y)} \tag{3a, b}$$

or alternatively

$$x=\frac{1}{2\sigma}\tan^{-1}\left(\frac{2\sigma u}{u^2+v^2-\sigma^2}\right) \qquad y=\frac{1}{2\sigma}\tanh^{-1}\left(\frac{2\sigma v}{u^2+v^2+\sigma^2}\right).$$

The (u, v) frame is illustrated in figure 3(a). The solid curves in figure 3(a) are the possible particle boundaries given by constant u, while the dashed lines are the

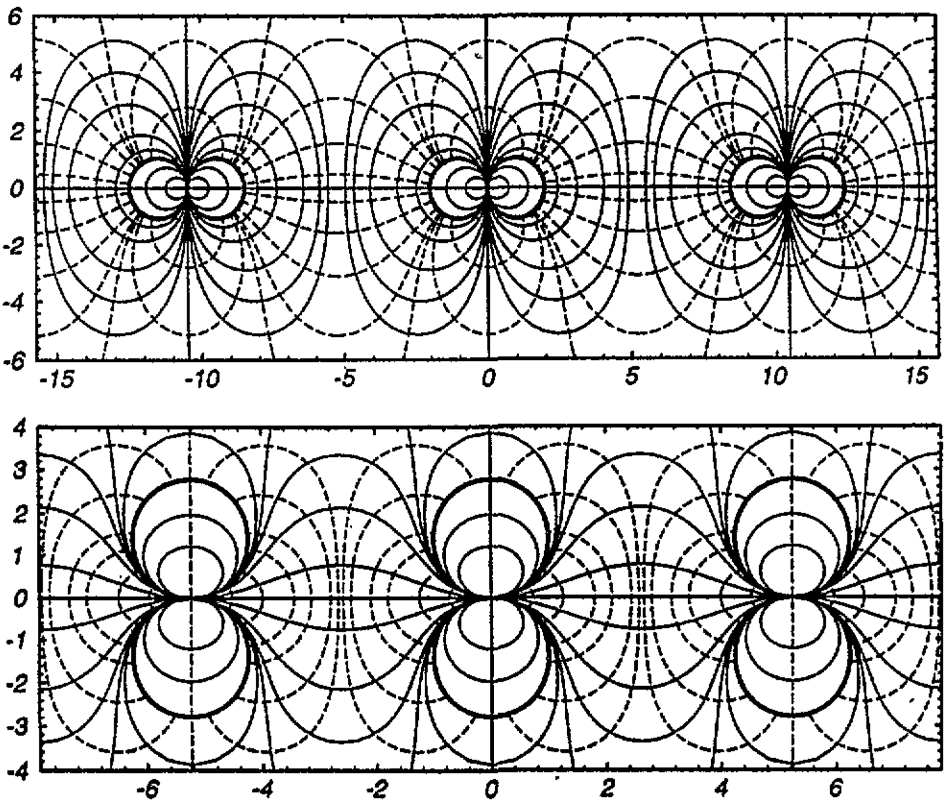

Figure 3. Three cells in the 'chain' coordinate frames. In both cases the lines of constant u (solid) and constant v (broken) are shown. The particle boundaries of chain 1 (a) and chain 2 (b) corresponding to a fill factor f of 0.3 are shown in bold.

orthogonal curves given by constant v. We shall define the central cell to be that for which $-a/2 \leqslant x \leqslant a/2$. The sign changes within this central cell for u and v are indicated in figure 4. It is clear from figure 3(a) that the contours of constant u in the (u, v) frame are not circles. In fact as the u value decreases the curves become less and less circular. However, for an appreciable range of u values (fill factor not large), the closed curves given by constant u *do* closely resemble circles.

In order to solve the boundary value problem for the electric potential it will eventually be necessary to decompose the applied field into its eigenfunction expansion. In fact, x and y will have to be expressed as integrals over the positive real axis. We shall therefore conclude this section by stating and proving the following theorem.

Theorem. Let $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$ and suppose there exists a function $\mathscr{F}$ such that

$$w = \mathscr{F}(\bar{z})$$

where the inverse function $\mathscr{F}^{-1}(w)$ is analytic in the domain $u > \gamma$ and tends to zero uniformly in v as $|w| \to \infty$. If the integral of $|\mathscr{F}^{-1}(w)|$ along any vertical line $u = \gamma' > \gamma$ is convergent, then for $\lambda > 0$

$$\mathscr{L}_\lambda^{-1}\{\mathscr{F}^{-1}(w)\} = \frac{1}{2\pi\mathrm{i}} \int_{\gamma'-\mathrm{i}\infty}^{\gamma'+\mathrm{i}\infty} \mathrm{e}^{\lambda w} \mathscr{F}^{-1}(w)\, \mathrm{d}w \qquad \gamma' > \gamma$$

exists [12], and

$$x = \int_0^\infty \mathscr{L}_\lambda^{-1}\{\mathscr{F}^{-1}(w)\}\, \mathrm{e}^{-\lambda u} \cos \lambda v\, \mathrm{d}\lambda \qquad y = \int_0^\infty \mathscr{L}_\lambda^{-1}\{\mathscr{F}^{-1}(w)\}\, \mathrm{e}^{-\lambda u} \sin \lambda v\, \mathrm{d}\lambda \tag{4a, b}$$

where $\mathscr{L}_\lambda$ is the *Laplace transform* in λ.

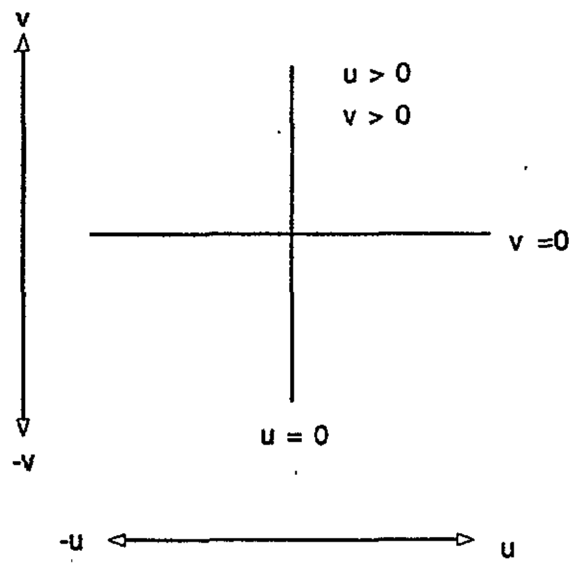

Figure 4. The sign changes within a single cell in the coordinate frame for chain 1.

Proof. Since

$$\bar{z} = \mathscr{F}^{-1}(w)$$
$$= \mathscr{L}_\lambda\{\mathscr{L}_\lambda^{-1}\{\mathscr{F}^{-1}(w)\}\}$$
$$= \int_0^\infty \mathscr{L}_\lambda^{-1}\{\mathscr{F}^{-1}(w)\}\, \mathrm{e}^{-\lambda w}\, \mathrm{d}\lambda$$
$$= \int_0^\infty \mathscr{L}_\lambda^{-1}\{\mathscr{F}^{-1}(w)\}\, \mathrm{e}^{-\lambda u} \cos \lambda v\, \mathrm{d}\lambda - \mathrm{i} \int_0^\infty \mathscr{L}_\lambda^{-1}\{\mathscr{F}^{-1}(w)\}\, \mathrm{e}^{-\lambda u} \sin \lambda v\, \mathrm{d}v$$

the result follows upon equating real and imaginary parts. Equations (4) thus represent alternative expressions for x and y in terms of the function $\mathscr{F}$.

2.2. *The general solution of the Laplace equation*

As the transformation (2) is conformal the Laplace equation in the (u, v) frame is [13]

$$\phi_{uu} + \phi_{vv} = 0$$

for the electric potential ϕ. For some separation constant λ, the partial solutions of the Laplace equation, $\phi(\lambda)$, will take one of the following forms:

$$(A_\lambda \cosh \lambda u + B_\lambda \sinh \lambda u)(C_\lambda \cos \lambda v + D_\lambda \sin \lambda v)$$

or

$$(A'_\lambda \cosh \lambda v + B'_\lambda \sinh \lambda v)(C'_\lambda \cos \lambda u + D'_\lambda \sin \lambda u).$$

As there are no restrictions on λ in this situation the eigenvalue spectrum will be continuous. The general solution for the potential is thus of the form [13, 14]

$$\phi = \int_{-\infty}^\infty \phi(\lambda)\, \mathrm{d}\lambda.$$

2.3. *The longitudinal polarizability for chain 1*

We now find an expression for the (longitudinal) polarizability of chain 1 resulting from a constant field applied along the x-axis. The potential for this applied field will be given by

$$\phi^{app} = -x. \tag{5}$$

The symmetry requirements on $\phi(u, v)$ in this situation are:

(i) $\phi(u, -v) = \phi(u, v)$ (even in y)

(ii) $\phi(-u, v) = -\phi(u, v)$ (odd in x).

For a pair of particles whose boundaries are given by $u = \pm u_1$, the following conditions apply:

$$\text{on } u = u_1 : \begin{cases} \dfrac{\partial}{\partial u}\{\phi^{app} + \phi^{(1)}\} = \varepsilon \dfrac{\partial}{\partial u}\phi^{(2)} \\ \phi^{app} + \phi^{(1)} = \phi^{(2)} \end{cases} \tag{6a, b}$$

where $\phi^{(1)}$ and $\phi^{(2)}$ denote the potentials external and internal to the particles respectively. Now, as region 2 includes the Cartesian origin ($u=\infty$), the third condition becomes

(iii) ϕ is bounded for large positive u.

Finally, the solution pair $\{\phi^{(1)}, \phi^{(2)}\}$ must also admit a so-called non-trivial *resonant solution* [5, 15], that is, a solution satisfying (6) with $\phi^{app}\equiv 0$. The fourth condition is then

(iv) There exists a non-trivial resonant solution.

The only non-trivial solution form satisfying the symmetry conditions ((i) and (ii)), the boundary conditions (6), the boundedness condition (iii), and which admits a resonant solution (iv), is given by

$$\phi^{(1)}=\int_0^\infty A_\lambda^{(1)} \sinh \lambda u \cos \lambda v \, \mathrm{d}\lambda \tag{7a}$$

and

$$\phi^{(2)}=\int_0^\infty A_\lambda^{(2)} \mathrm{e}^{-\lambda u} \cos \lambda v \, \mathrm{d}\lambda. \tag{7b}$$

It can thus be seen that the eigenfunctions of this problem are

$$\mathrm{e}^{\pm\lambda u} \cos \lambda v \qquad \lambda \in \{0\} \cup \boldsymbol{R}^+.$$

The coefficients $A_\lambda^{(i)}$ will then be found by substituting (5) and (7) into (6). It therefore only remains to decompose ϕ^{app} into its eigenfunction expansion in u and v. Now from (2) we have

$$\mathscr{F}(\bar{z})=\sigma \cot \sigma\bar{z}$$

which satisfies the conditions of the *theorem* with $\gamma=0$. Moreover since [16]

$$\mathscr{L}_\lambda^{-1}\left\{\frac{1}{\sigma}\cot^{-1}\frac{w}{\sigma}\right\}=\frac{\sin \lambda\sigma}{\lambda\sigma}$$

the following expression for x is obtained upon applying (4a):

$$x=\int_0^\infty \frac{\sin \lambda\sigma}{\lambda\sigma} \mathrm{e}^{-\lambda u} \cos \lambda v \, \mathrm{d}\lambda. \tag{8}$$

Equation (8) represents the decomposition of x in terms of the eigenfunctions $\mathrm{e}^{-\lambda u} \cos \lambda v$. Substitution of (5), (8) and (7) into (6) now yields the following expression for the external potential:

$$\phi^{(1)}=-\int_0^\infty \frac{2\tau \sin \lambda\sigma}{\lambda\sigma(\tau+\mathrm{e}^{2\lambda u_1})} \sinh \lambda u \cos \lambda v \, \mathrm{d}\lambda \qquad \tau=\frac{1-\varepsilon}{1+\varepsilon} \tag{9}$$

where τ is called the dielectric contrast. The boundary conditions (6) lead to a resonant solution given by

$$\tanh \lambda_0 u_1=-1/\varepsilon$$

which has the same form as the corresponding result in [2] for an isolated cylinder pair. The difference here is that u_1, and thus λ_0, will depend on the fill factor.

In order to find the polarizability we now expand the external potential $\phi^{(1)}$ along the x-axis ($v=0$) to $O(1/r)$. By setting $y=0$ in (3a) it is easily seen that along the x-axis

$$u=\sigma \cot \sigma x.$$

For large $x>0$ we let $x=r+na$, $n \in N$, $|r|<a/2$, and since

$$\cot \sigma(r+na)=\cot(\sigma r+n\pi)=\cot \sigma r$$

we are required to find the coefficient of $1/r$ in the following:

$$-2\tau \int_0^\infty \frac{\sin \lambda\sigma \sinh(\lambda\sigma \cot \sigma r)}{\lambda\sigma(\tau+\mathrm{e}^{2\lambda u_1})}\,\mathrm{d}\lambda \qquad r<a/2. \tag{10}$$

By expanding the function $\sin \lambda\sigma$ and expressing the remaining integral as a series [17], we find

$$\phi^{(1)}_{(v=0)}=\sum_{n=0}^\infty \sum_{k=1}^\infty \frac{(-1)^n\sigma^{2n}}{2n+1}\left[\frac{(-\tau)^k}{\{2u_1k-\sigma \cot \sigma r\}^{2n+1}}-\frac{(-\tau)^k}{\{2u_1k+\sigma \cot \sigma r\}^{2n+1}}\right] \tag{11}$$

as the potential along the x-axis. Expansion of the denominators in (11) will yield integer powers of $\cot \sigma r$. Hence we need to find the coefficient of $1/r$ in all positive integer powers of $\cot \sigma r$. We shall do this by finding the residue of $\cot^n z$, $n=1, 2, \ldots$.

By noting that this residue is given by

$$\int_{|z|=1} \cot^n z\, \mathrm{d}z$$

it is easily shown using integration by parts that

$$\mathrm{Res}\{\cot^n z\}=\begin{cases}(-1)^m & n=2m+1\\ 0 & n=2m.\end{cases} \tag{12}$$

We now use the binomial theorem to expand the denominators in (11) and then apply (12) to extract the coefficient of $1/r$ from the resulting expression. When the remaining series over n is then put in closed form we are left with

$$\frac{\mathrm{i}}{\sigma^2}\sum_{k=1}^\infty (-\tau)^k\left[\tan^{-1}\left(\frac{\sigma}{2u_1k+\mathrm{i}\sigma}\right)-\tan^{-1}\left(\frac{\sigma}{2u_1k-\mathrm{i}\sigma}\right)\right]$$

as the $1/r$ coefficient in (11). Further algebra leads to the following form for the longitudinal polarizability $\alpha_\parallel$:

$$\alpha_\parallel=\frac{2\pi}{\sigma^2}\sum_{k=1}^\infty (-\tau)^k \tanh^{-1}\left(\frac{\sigma^2}{2u_1^2k^2+\sigma^2}\right). \tag{13}$$

In the limit of large separation between particle pairs ($\sigma \to 0$) the polarizability (13) should reduce to that for an isolated pair of circles of radius R. We now scale all dimensions to this radius, thus effectively setting $R=1$. The chain parameter σ is specified by f, the fill factor of metal in the film matrix, while u_1 is chosen so that the particle pair in the central cell cuts the x-axis at the origin and at the points ± 2 (see appendix I(a)). These requirements lead to

$$\sigma=f \qquad u_1=f \cot 2f. \tag{14}$$

By using (14) to re-express (13) in terms of f we obtain the following:

$$\alpha_\parallel=\frac{\pi}{f^2}\sum_{k=1}^\infty (-\tau)^k \log\left(1+\frac{\eta^2}{k^2}\right) \qquad |\tau|<1 \tag{15}$$

where $\eta = \tan 2f$. At this stage we may note that in the limit of small f (15) reduces to the familiar dilogarithm formula for the polarizability of an isolated pair of touching unit circles [2, 15].

As the dielectric contrast τ is in general a complex number whose modulus may be greater than unity, we must find the analytic continuation of the series in (15) in order to arrive at a uniformly valid expression for $\alpha_{\|}$. To this end we now perform the following sequence of operations. Differentiating the series in (15) with respect to η, writing the result as a sum of partial fractions, and then integrating with respect to η yields

$$\alpha_{\|} = \frac{\mathrm{i}\pi\tau}{f^2} \int_0^{\eta} [\Phi(-\tau, 1, 1-\mathrm{i}t) - \Phi(-\tau, 1, 1+\mathrm{i}t)]\,\mathrm{d}t \tag{16}$$

where

$$\Phi(z, s, v) = \sum_{n=0}^{\infty} \frac{z^n}{(v+n)^s} \qquad |z|<1, v \neq 0, -1, -2, \ldots$$

is the *Lerch transcendent.* Using the fact that [18]

$$\Phi(z, s, v) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}\,\mathrm{e}^{-vt}}{1-z\,\mathrm{e}^{-t}}\,\mathrm{d}t$$

$$(\mathrm{Re}\, v>0 \text{ and either } |z| \leqslant 1, z \neq 1, \mathrm{Re}\, s>0 \text{ or } z=1, \mathrm{Re}\, s>1)$$

the integrand in (16) can be re-expressed as

$$2\mathrm{i} \int_0^{\infty} \frac{\sin \mu t}{\mathrm{e}^{\mu}+\tau}\,\mathrm{d}\mu$$

yielding the following expression for the longitudinal polarizability

$$\alpha_{\|} = -\frac{4\pi\tau}{f^2} \int_0^{\infty} \frac{\sin^2[(\mu \tan 2f)/2]}{\mu(\mathrm{e}^{\mu}+\tau)}\,\mathrm{d}\mu \tag{17a}$$

which is valid for all complex τ in the cut plane and thus represents the analytic continuation of (15).

2.4. The transverse polarizability for chain 1

We now present the main results in the derivation of the transverse polarizability. For a field applied in the y-direction a unique solution pair $\{\phi^{(1)}, \phi^{(2)}\}$ can be found using a sequence of steps analogous to those above. A non-trivial resonant solution exists, in this case, for the eigenfunctions

$$\mathrm{e}^{\pm\lambda u} \sin \lambda v \qquad \lambda \in \{0\} \cup \boldsymbol{R}^{+}.$$

By using (4b) the applied field potential

$$\phi^{app} = -y$$

is decomposed into its eigenfunction expansion. Satisfaction of the boundary equations (6) and of the relevant symmetry and boundedness conditions then leads to the external potential

$$\phi^{(1)} = \int_0^{\infty} \frac{2\tau \sin \lambda\sigma}{\lambda\sigma(\mathrm{e}^{2\lambda u_1}-\tau)} \cosh \lambda u \sin \lambda v\,\mathrm{d}\lambda$$

with resonant solution

$$\tanh \lambda_0 u_1 = \varepsilon$$

the latter again coinciding in its form with the corresponding result in [2]. We use (3b) this time to determine $\phi^{(1)}$ along the y-axis, and then find the polarizability from the coefficient of $1/r$ in the expansion of

$$2\tau \int_0^\infty \frac{\sin \lambda\sigma}{\lambda\sigma(e^{2\lambda u_1} - \tau)} \sin(\lambda\sigma \coth \sigma r)\, d\lambda$$

which turns out to be

$$\frac{-2\pi}{\sigma^2} \sum_{k=1}^{\infty} \tau^k \tanh^{-1}\left(\frac{\sigma^2}{2u_1^2 k^2 + \sigma^2}\right).$$

Proceeding as in section 2.3, it can be shown that the transverse polarizability, $\alpha_\perp$, in terms of the fill factor f, is given by

$$\alpha_\perp = \frac{-4\pi\tau}{f^2} \int_0^\infty \frac{\sin^2[(\mu \tan 2f)/2]}{\mu(e^\mu - \tau)}\, d\mu. \tag{17b}$$

3. Polarizabilities for chain 2

As can be seen from figure 3 the boundaries of particles in chain 2 are precisely the orthogonal curves of the coordinate frame for chain 1. Hence the transformation for this case is essentially the same as in section 2.1, namely:

$$w = i\sigma \cot \sigma z.$$

Here, curves given by constant $|u|$ are pairs of closed curves whose points of tangency all lie along the x-axis at na, $n \in Z$, but whose axis this time is *perpendicular* to the x-axis (see figure 3(b)).

A sequence of steps mirroring those used in section 2 is followed, leading ultimately to the following expressions for the longitudinal and transverse polarizabilities for chain 2:

$$\alpha_{\parallel,\perp} = -\frac{\pi\tau}{f^2} \int_0^\infty \frac{\sinh^2[(\mu \sin 4f)/2]}{\mu(e^\mu \mp \tau)}\, d\mu \tag{18a, b}$$

where $\sigma = 2f$ and $u_1 = 2f \operatorname{cosec} 4f$. Here, as for chain 1, σ is found from the fill factor f and u_1 is chosen so that the particle pair lies within the vertical strip given by $-1 \leqslant x \leqslant 1$ (see appendix I(b)). We conclude this section by noting that both (17) and (18) satisfy Keller's [19] reciprocal relationship.

4. Results and discussion

The polarizabilities are determined from equations (17) and (18) and then normalized by dividing by the area of the respective particle pair (see appendix II). Using data for aluminium columns in air, the real and imaginary parts of the normalized polarizabilities are graphed as functions of wavelength. In order to facilitate the interpretation of results, the three responses (for the two chains and the isolated pair

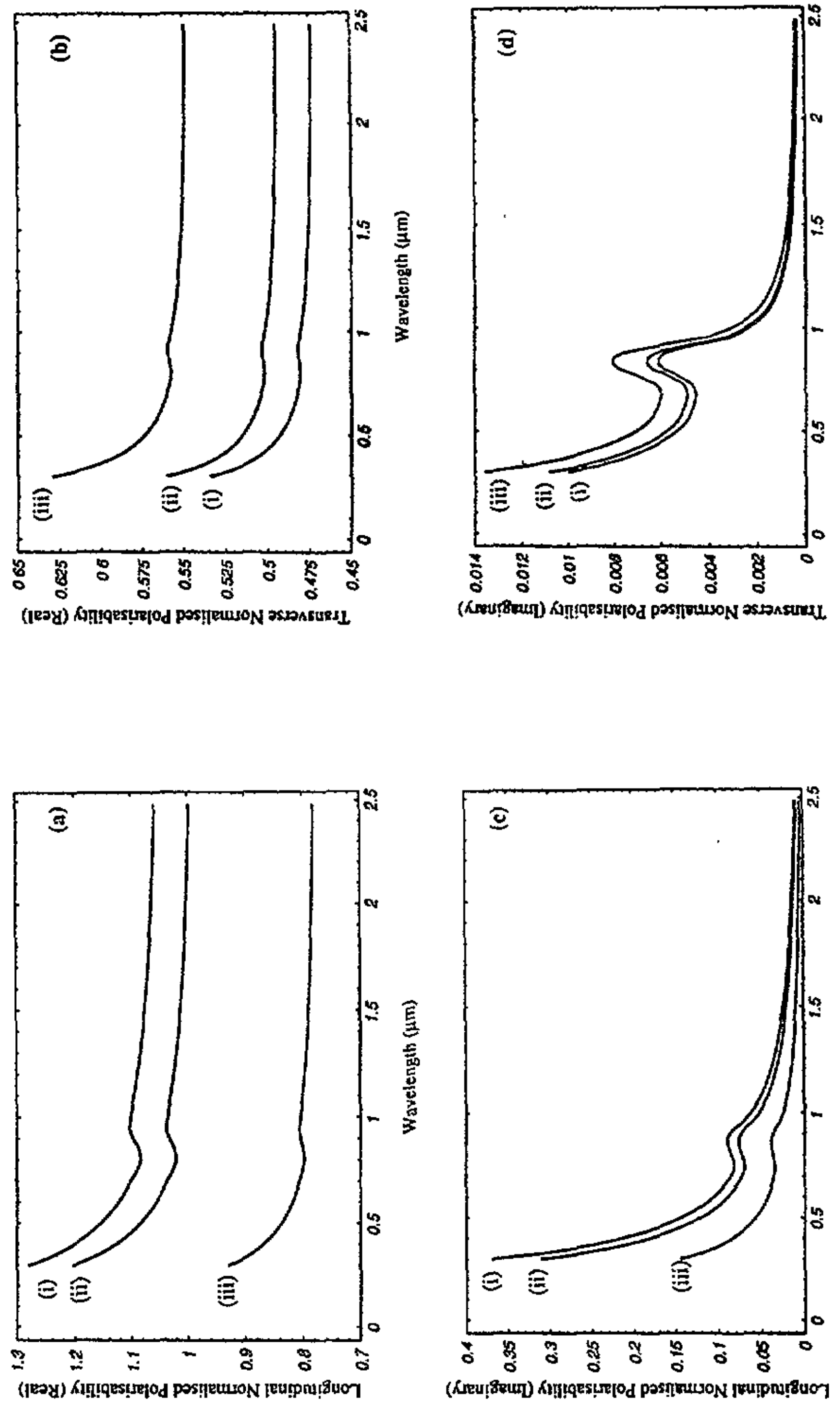

Figure 5. The real and imaginary parts of the normalized polarizability as a function of wavelength for aluminium columns in air. The real parts of the longitudinal and transverse polarizabilities for (i) chain 1, (ii) the single pair and (iii) chain 2 are graphed in (*a*) and (*b*) respectively, the corresponding imaginary parts in (*c*) and (*d*) respectively.

of touching unit circles) are all graphed together according to the following scheme. We shall denote by *longitudinal* all those polarizabilities resulting from applied field orientation/particle type combinations appearing in figure 1. In other words, here longitudinal will mean an applied field which is parallel to the axis of the particle pair. Similarly the *transversal* polarizabilities will be those corresponding to applied field orientation/particle type combinations appearing in figure 2. So here, transversal will signify an applied field perpendicular to the axis of the particle pair. Results are computed for $f=0.3$ throughout figure 5. Figures 5(a) and 5(b) show the real parts of the normalized polarizability for the longitudinal and transverse fields while figures 5(c) and 5(d) show the corresponding imaginary parts.

The graphs are physically reasonable and are consistent with what one would expect to find. For any direction of the applied field $\boldsymbol{E}$ there are two factors contributing to the total polarization of each particle. The first and principle contribution is the degree of polarization which would result were the particle isolated. The second contribution, of order f^2, is due to the combined polarization effective fields set up between particles to the left and to the right of each given particle.

Consider now the *longitudinal* cases, that is, those illustrated in figure 1. For chain 1 (figure 1(a)) the two components contributing to the polarization are parallel and so there is a relative enhancement in polarizability over that of an isolated pair. In chain 2 however (figure 1(c)), the repulsion of the polarization charge between neighbouring parallel cells results in a redistribution of the charge over the edges of each particle in such a way that the total polarization decreases relative to that of an isolated pair (figure 1(b)). The situation obtaining in the *transversal* cases (figure 2) will be the converse of the above. So, relative to the transversal polarization of an isolated particle pair (figure 2(b)), that for chain 1 will be diminished (figure 2(a)) whilst that for chain 2 will be enhanced (figure 2(c)). In both cases though, the long-range interactions are smaller in comparison to the polarization of an isolated pair.

5. Conclusion

In this work we have solved the two-dimensional electrostatic problem for two different chains of infinite extent. The fundamental particle in each case is a pair of touching closed curves whose axis may be aligned along (chain 1) or at right angles to (chain 2) the chain axis. *Exact* analytic expressions (17), (18) have been derived for the polarizability of the particle pairs in each chain. The method involves the construction, through a conformal mapping, of the appropriate 'chain' coordinate frame within which the Laplace equation is subsequently solved. The technique of finding the $1/r$ coefficient of a real function by finding the residue of the corresponding complex function has been found very useful in the determination of the polarizability.

Working at a fill factor of 0.3 we have calculated the real and imaginary parts of the normalized polarizability for the two chains as a function of the applied field wavelength. The resulting curves have been compared with the corresponding ones for an isolated pair of touching unit circles (figure 5). We see that, at least for small fill factors, the polarization will be largely accounted for by the single isolated pair model. Nonetheless, for certain orientations of the applied field the absorption for the chains is noticeably greater than that for the isolated pair (figure 5(c), (d)). In fact, it has been found that when the fill factor is increased to 0.5 these enhancements are even larger. Hence we are led to conclude that when high fill factors prevail the long range effects become important. In this situation then the isolated pair model can be expected to seriously underestimate the degree of absorption.

Although only strictly applicable at relatively low fill factors, this model has signalled the importance of longer range effects for close approach between the cylindrical inclusions. In light of this conclusion, it would therefore be worthwhile developing a model permitting study of cylinders which retain their circularity even at high fill factors. Such work would confirm in detail the quantitative predictions of the present model.

Appendix

Appendix I

In the following we derive expressions for the chain parameters σ and u_1 in terms of the fill factor f.

(*a*) For chain 1 we begin by considering the infinite chain of pairs of touching circles shown in figure A1. We shall consider a fundamental cell to be the rectangle $ABCD$ with semi-base length L. For circles of radius R the cell area will be $4LR$. The fill factor f will be defined as

$$\frac{\text{Area of particle pair}}{\text{Area of fundamental cell}}.$$

Hence

$$f=\frac{\pi R}{2L}.$$

All lengths are now scaled by taking the unit of length to be R (effectively $R=1$) and so

$$L=\frac{\pi}{2f}. \tag{i}$$

In chain 1

$$L=\frac{a}{2}=\frac{\pi}{2\sigma}. \tag{ii}$$

Eliminating L between (i) and (ii) yields

$$\sigma=f. \tag{iii}$$

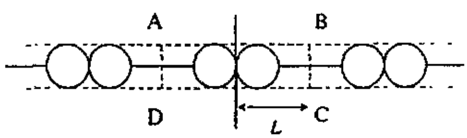

Figure A1. An infinite chain of pairs of touching circles with pair axis along the chain axis. The semi-base length of the fundamental cell *ABCD* is *L*.

The value of u_1 is found by requiring that the pair of closed curves in chain 1 (given by constant $|u_1|$ and shown in bold in figure 3(a)) cuts the x-axis at the origin and at the points ± 2. So, using (3a)) with $u=u_1$, $x=2$ and $y=0$ we have:

$$u_1=\frac{\sigma \sin 4\sigma}{2\sin^2 2\sigma}=\sigma \cot 2\sigma. \tag{iv}$$

Together (iii) and (iv) imply

$$u_1=f\cot 2f.$$

(b) For chain 2 we consider the infinite chain of pairs of touching unit circles shown in figure A2. By defining the fill factor as in (a) we find that here

$$f=\frac{\pi}{4L}$$

and so

$$\sigma=2f. \tag{v}$$

To find u_1 this time we shall require that the particle pair in chain 2 (given by constant $|u_1|$ and shown in bold in figure 3(b)) lies between, and tangent to, the pair of vertical lines given by $x=\pm 1$. This situation is shown below in figure A3. The points E and F lie on $x=1$, the points G and H on $x=-1$. For chain 2

$$u=\frac{\sigma \sinh 2\sigma y}{2(\sin^2 \sigma x+\sinh^2 \sigma y)}. \tag{vi}$$

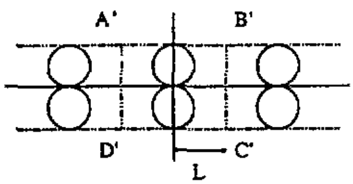

Figure A2. An infinite chain of pairs of touching circles with pair axis perpendicular to the chain axis. The semi-base length of the fundamental cell $A'B'C'D'$ is L.

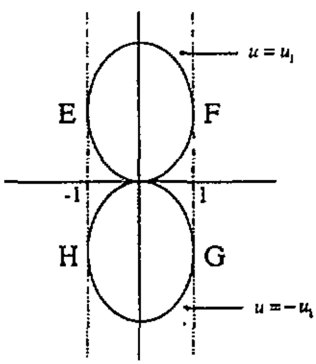

Figure A3. A particle pair from chain 2 with bounding curves given by $u=\pm u_1$. The parameter u_1 is chosen so that these curves are tangent to the vertical lines $x=\pm 1$ at the points E, F, G and H.

Setting $dx/dy = 0$ in (vi) with $u = u_1$ leads to

$$\tanh 2y = \frac{\sigma}{u_1}. \tag{vii}$$

Eliminating σ and y from (vi) by means of (v) and (vii) respectively, and then setting $x = 1$ in the result, leads to

$$u_1 = 2f \operatorname{cosec} 4f.$$

Appendix II

In the following ${}_2F_1(a, b; c; z)$ will denote the *hypergeometric series* of the variable z with parameters a, b, and c [18]. The results shown in figure 5 are polarizabilities normalized with respect to area. For chain 1

$$\bar{\alpha}_{\|,\perp} = -\frac{4\pi\tau}{N_1 f^2} \int_0^\infty \frac{\sin^2[(\mu \tan 2f)/2]}{\mu(e^\mu \pm \tau)} \, d\mu$$

where

$$N_1 = \frac{4}{f^2} \int_0^2 \sinh^{-1}\{\sin fx \sqrt{\tan 2f \cot fx - 1}\} \, dx.$$

Calculations were checked by means of the relation

$$\int_0^\infty \frac{\sin^2[(\mu \tan 2f)/2]}{\mu(e^\mu \pm \tau)} \, d\mu = \frac{1}{4} \int_0^{\tan 2f} \frac{{}_2F_1(1, 1-\mathrm{i}t, 2-\mathrm{i}t, \mp\tau)}{t+\mathrm{i}} + \frac{{}_2F_1(1, 1+\mathrm{i}t, 2+\mathrm{i}t, \mp\tau)}{t-\mathrm{i}} \, dt.$$

For chain 2,

$$\bar{\alpha}_{\|,\perp} = -\frac{4\pi\tau}{N_2 f^2} \int_0^\infty \frac{\sinh^2[(\mu \sin 4f)/2]}{\mu(e^\mu \mp \tau)} \, d\mu$$

where

$$N_2 = \frac{8}{f^2} \int_0^1 \tanh^{-1} \sqrt{1 - \frac{\cos^2 4f}{\cos^2 4fx}} \, dx$$

which is checked using the relation

$$\int_0^\infty \frac{\sinh^2[(\mu \sin 4f)/2]}{\mu(e^\mu \mp \tau)} \, d\mu = \frac{1}{4} \int_0^{\sin 4f} \frac{{}_2F_1(1, 1+t, 2+t, \pm\tau)}{t+1} + \frac{{}_2F_1(1, 1-t, 2-t, \pm\tau)}{t-1} \, dt.$$

All numerical computations were carried out using *Mathematica* [20].

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